ON THE STABILITY OF POISEUILLE FLOW AND CERTAIN OTHER PLANE-PARALLEL FLOWS IN A FLAT PIPE OF LARGE BUT FINITE LENGTH FOR LARGE REYNOLDS' NUMBERS

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The stability of plane-parallel flows in a flat pipe of large but finite length is studied for large Reynolds' numbers on the basis of notions about the stability of homogeneous states [1]. It is shown that plane-parallel flows with a convex symmetrical unperturbed velocity profile are not globally unstable. An example of a globally unstable flow with a velocity profile containing inflection points is constructed.

Let us consider the steady-state flow of a viscous incompressible fluid in a pipe of constant cross-section and large length $-L \leqslant x \leqslant L$. The Reynolds' number computed over the width of the channel will be assumed sufficiently large. The velocity profile can be assumed Poiseuillian and independent of x everywhere except the segments of finite length near the ends of the pipe. This region will be referred to as the principal part of the flow. We shall assume that certain time-independent boundary conditions have been specified at the pipe ends $x = \pm L$. These boundary conditions interrelate the perturbations of the hydrodynamic quantities and their derivatives and the boundary conditions at each end include the values of the indicated quantities at that end. An example of this is the condition that the velocity perturbation is equal to zero at $x = \pm L$. This condition can be considered fulfilled if the fluid flows in and out through porous walls at the pipe ends.

An important factor in the formulation of the problem is the inflow and outflow of the fluid through the boundaries of the region under consideration.

Since the unperturbed state and conditions at the boundaries are steady, the dependence of the perturbations of velocity and pressure on time is given by the factor $\exp(-i\omega t)$, and the problem consists in finding the eigenvalues of ω. In the principal part of the flow for a given ω the dependence of the perturbations on x is given by the factor $\exp ikx$, where for each ω the permissible values of k are determined from the condition of existence of a nontrivial solution of the boundary value problem in the yz-plane perpendicular to the pipe axis (the x-axis). In the case of a flow between two planes, the value of k must be chosen in such a way that there exists a nontrivial solution of the Orr-Sommerfeld equation (e.g. see [2])

$$\left(\frac{d^2}{dy^2}-k^2\right)^2\varphi = ikR\left[\left(u-\frac{\omega}{k}\right)\left(\frac{d^2}{dy^2}-k^2\right)\varphi - \frac{d^2U}{dy^2}\varphi\right] \qquad (0.1)$$

subject to the boundary conditions

$$\varphi(\pm 1)=0, \qquad \left.\frac{d\varphi}{dy}\right|_{y=\pm 1}=0 \qquad (0.2)$$

where y is a dimensionless coordinate, $U(y)$ is the unperturbed velocity, and $\varphi(y)$ is a function related to the stream function $\psi(x, y, z, t)$ for the velocity perturbation, by the expression

$$\psi(x, y, z, t) = \varphi(y) \exp i(kx - \omega t)$$

Since the coefficient of the highest-order derivative in Equation (0.1) is constant, while the other coefficients depend on ω and k analytically and do not become infinite for finite values of ω and k, it follows that the solutions of Equation (0.1) depend on ω and k analytically. Hence, the condition for the existence of a nontrivial solution of the problem (0.1) and (0.2), is of the form

$$G(\omega, k) = 0 \qquad (0.3)$$

where G is an analytic meromorphic function of ω and k expressed in terms of the values for $y = \pm 1$ of the functions making up the fundamental system of solutions of Equation (0.1), and their derivatives with respect to y taken at the same points.

By virtue of Equation (0.3), each ω generally has infinitely many corresponding values $k_j(\omega)$, which are the eigenvalues of boundary value problem (0.1) and (0.2). Functions $k_j(\omega)$ are the branches of the analytic function $k(\omega)$ which results from the solution of Equation (0.3). Each branch $k_j(\omega)$ corresponds to some wave of the form $\exp i[k_j(\omega)x-\omega t]$ which propagates along the pipe. The conditions at the ends of the pipe together with the transitional zones (where the Poiseuille's velocity profile is established) generate certain effective boundary conditions which, for $x = \pm L$, relate the amplitudes of the waves propagating in the principal part of the flow from the pipe ends.

Our problem concerning the stability of flow in a sufficiently long flat pipe belongs to the category of problems considered in [1]. In fact, the dependence $k(\omega)$ obtained from (0.3) is such, that for sufficiently large $\operatorname{Im}\omega$, real values of k which would satisfy Equation (0.3), do not exist [2].

We shall assume that the boundary conditions for $x = \pm L$ guarantee the correctness of the problem's formulation. This assumption is valid if, for example, for $x = \pm L$ the velocity perturbation vanishes, since in this case the solution of the plane unsteady state problem of fluid flow exists, and is unique [3].

In [1] it is shown that for sufficiently large L the characteristic solutions which depend on time as $\exp(-i\omega t)$ are reducible to two types – the unilateral and the global. The complex frequency ω corresponding to unilateral solutions depends on the actual form of the boundary conditions at one of the ends. In global solutions, the limit of ω as $L \to \infty$ does not depend on the actual form of the boundary conditions. The global characteristic solutions are analogous to the quasiclassical solutions employed in the study of weakly nonhomogeneous systems [4].

For global solutions, the characteristic frequency with the largest imaginary part is generally the solution of the equation

$$\mathrm{Im}\,[k_s(\omega) - k_{s+1}(\omega)] = 0 \qquad (0.4)$$

possessing the largest imaginary part. Let us denote this solution by ω_*. The functions $k_s(\omega)$ and $k_{s+1}(\omega)$ in Equation (0.4) denote the branches of the function $k(\omega)$ for which $\mathrm{Im}(\omega) > 0$ and $\mathrm{Im}\, k_{s+1}(\omega) < 0$ for sufficiently large $\mathrm{Im}\,\omega$, and which yield the solution of Equation (0.4)) with the largest imaginary part. The eigenfunction with $\omega = \omega_*$ represents a self-maintaining system of waves corresponding to $k_s(\omega)$ and $k_{s+1}(\omega)$. The condition under which the value ω_* is associated with a certain eigenfunction of the problem under consideration is,that the boundary conditions for $x = \pm L$ guarantee the reflection of the $(s+1)$ th and s-th waves with amplitudes not identically equal to zero in ω when the s-th and $(s+1)$-th waves are incident on $x = L$ and $x = -L$, respectively. This condition is, usually always fulfilled, and it can always be achieved by a slight alteration of the initial conditions. Finally, in the exceptional case when this condition is not fulfilled for the s-th and $(s+1)$-th waves, the global eigenfunctions can be formed on the basis of other wave pairs for which the corresponding branches $k_i(\omega)$ and $k_j(\omega)$ are subject to the condition $\mathrm{Im}\, k_i(\omega) > 0$ and $\mathrm{Im}\, k_j(\omega) < 0$ with $\mathrm{Im}\,\omega$ sufficiently large. The resulting values of ω obviously have an imaginary part which is smaller than ω_*. Hence, if Equation (0.4) has no solutions with $\mathrm{Im}\,\omega > 0$, then there is no global instability.

We note that the application of the results of [1] to systems in which infinitely many k_j correspond to each ω is based on the assumption of a possible limiting process wherein the boundary value problem for a system with an infinite number of waves corresponding to the prescribed ω is obtained as the limit of a sequence of correct boundary value problems for systems with a finite number of waves. Although such a limiting process for Poiseuille flow was not considered, the possibility of its realization appears entirely plausible. In addition, the existence of an eigenfunction corresponding to the solution of Equation (0.4) follows from the physical meaning of the phenomenon, independent of the total number of waves, involved in the passage and reflection of the s-th and $(s+1)$-th wave [1]. It is therefore interesting to investigate Equation (0.4) in connection with the problem of the stability of Poiseuille flow.

As usual, along with the problem of the stability of Poiseuille flow, we shall also consider the stability of plane-parallel flows characterized by a differently specified unperturbed velocity $U(y)$. We note that flows of a viscous fluid with a non-Poiseuille velocity profile in the absence of external forces can only be approximately considered plane-parallel, so that some caution is required in the interpretation of the obtained results.

It will be shown below that if the unperturbed velocity profile is symmetrical $(U(-y) = U(y))$ and convex $(U''(y) > 0$ for all $y)$ and, if the Reynolds' number is sufficiently large, then Equation (0.4) has no roots in the upper half-plane ω. We shall later show an uperturbed velocity profile with inflection points such that the corresponding flow is globally unstable.

The presence of instability in the sense in which we understand it in the present study leads to an unlimited rise of perturbations in the flow, so that laminar flow becomes impossible; specifically, there cannot be a laminar flow segment at the pipe entrance.

1. On the complex plane ω, let us consider the half-plane $\mathrm{Im}\ \omega > b$ and examine the region Q (which may consist of several isolated parts) on the plane k, consisting of all points onto which the points of the half-plane $\mathrm{Im}\ \omega > b$ are mapped by means of at least one of the branches $k_j(\omega)$ of the analytic function $k(\omega)$ given implicitly by Equation (0.3). Boundary of the region Q consists of curves which represent mappings of the straight line $\mathrm{Im}\ \omega = b$ onto the plane k. According to [2], with sufficiently large values of b the real axis of k does not belong to these regions. One can also say that for arbitrarily large b there exist points which belong to Q and lie in both, the upper $\mathrm{Im}\ k > 0$, and the lower $\mathrm{Im}\ k < 0$ half-planes k. It can be shown, for example, that if $\omega \to \infty$ in the upper half-plane, and if k remains bounded, then $k \to i\pi n/2$, where $n \neq 0$ is an integer (positive or negative). These values of k correspond to the eigenfunctions of the Laplace equation

$$\varphi'' - k^2\varphi = 0$$

to which the equation for 'nonviscous' [2] solutions reduces as $\omega \to \infty$. Thus, for a sufficiently large b, the region Q is separated into two parts – the upper part Q_+ and the lower part Q_- which are not connected to each other.

As b diminishes the region Q can only expand, since the corresponding region on the plane ω also expands. If for $b = 0$ the regions Q_+ and Q_- on the plane k remain separated by a certain strip parallel to the real axis k, then Equation (0.4) does not have solutions in the upper half-plane ω. In fact, by virtue of the condition that the inequalities $\mathrm{Im}\ k_s > 0$ and $\mathrm{Im}\ k_{s+1} < 0$ are fulfilled for sufficiently large b, and as a consequence of the continuous dependence of k on ω, we can say that for any ω from the upper half-plane the roots k_s and k_{s+1} belong to Q_+ and Q_- respectively, so that Equation (0.4) in our case is not fulfilled for ω with $\mathrm{Im}\ \omega > 0$. The subsequent portion of the present study will be a proof of the fact that the regions Q_+ and Q_- on the plane k defined for $b = 0$ (and denoted by $Q_+{}^*$ and $Q_-{}^*$) by Equation (0.3), are for sufficiently large R divided by a strip parallel to the real axis of k.

2. Let us consider the curves $k_j(\omega)$ ($\mathrm{Im}\ \omega = 0$) which represent the mapping of the real axis of ω on the plane k. These curves can serve as the boundaries of the regions $Q_+{}^*$ and $Q_-{}^*$. We note first of all that if some $\omega = \alpha + ib$ has a corresponding $k = \alpha + i\beta$, then $k = -\alpha + i\beta$ corresponds to the value $\omega = -\alpha + ib$, since for these values boundary value problem (0.1), (0.2) has an eigenfunction which is the complex conjugate of the initial function. Hence, we shall be concerned from now on only with those portions of the curves $k_j(\omega)$ which correspond to $\omega > 0$ (we denote these by $k_j{}^*(\omega)$), since the portions of the curves $k_j(\omega)$ corresponding to $\omega < 0$ are in symmetry with them with respect to the imaginary axis of k.

As we know [2 and 5], only one branch $\omega_1(k)$ of the function $\omega(k)$, exists which assumes values with $\mathrm{Im}\ \omega > 0$ for real $k > 0$. The values of k for which $\mathrm{Im}\ \omega_1 > 0$ fill the segment $[k_1, k_2]$ on the real axis of k, outside which $\mathrm{Im}\ \omega_1(k) < 0$. Hence it follows, that the curves $k_j{}^*(\omega)$ intersect the real axis at the two points k_1 and k_2. In [6] it is shown that for large Reynolds' numbers, both of these points correspond to the intersection of the real axis with the same curve, which we denote by $k_*^{*}(\omega)$. The portion of this curve situated between the points k_1 and k_2 is situated in the lower half-plane, while the remaining portion lies in the upper half-plane.

The quantity $\operatorname{Im}\omega_1(k)$ is positive and bounded on the segment $[k_1, k_2]$ of the real axis of k and negative outside this segment [2]; it is evident, therefore, that the curve $k_*^{*}(\omega)$ represents the boundary of the region Q_+^{*} in the lower half-plane k. For flows with a convex unperturbed velocity profile, the values $k_*^{*}(\omega)$ with $\operatorname{Im} k_+^{*}(\omega) < 0$, the associated values of ω and $c = \omega/k$, simultaneously tend to zero as $R \to \infty$, although $k_*^{*} R$ and $R \operatorname{Im} k_*^{*}$ for the same values of $k_*^{*}(\omega)$ tend to infinity. Since, with the exception of $k_*^{*}(\omega)$, no other curves $k_*^{*}(\omega)$ intersect the real axis of k, region Q_-^{*} and the curve $k_l^{*}(\omega)$ which forms its upper boundary lie in the lower half-plane. It is clear that if Equation (0.3) has a solution ω with $\operatorname{Im}\omega > 0$ for large R, then $\operatorname{Im} k_{s+1} \to 0$ as $R \to \infty$. Let us therefore consider the branches of the function $k(\omega)$ which satisfy this condition.

3. Assuming the Reynolds' number to be sufficiently large, let us consider in more detail the relationship $k(\omega)$ which assures the existence of a nontrivial solution of boundary value problem (0.1) and (0.2). Since R appears in Equation (0.1) in the form of the combination kR, we shall consider only those values of k for which $kR \gg 1$.

In proving the aforementioned statement that the regions Q_+^{*} and Q_-^{*} are separated by a strip parallel to the real axis of k, we need not consider the small region $k \sim 1/R$, since the latter cannot contain the closed portion of the region Q^{*}. The latter statement follows from the fact that the region $k \lesssim 1/R$ has no points corresponding to $\omega = \infty$ ($k \to i\pi n/2$, $n \neq 0$ as $\omega \to \infty$ in the upper half-plane).

As we know, [2 and 7], boundary conditions (0.2) for $y = -1$ for sufficiently large kR can be written as follows:

$$C_1\varphi_1(-1) + C_2\varphi_2(-1) + C_3\varphi_3(-1) = 0 \tag{3.1}$$

$$C_1\varphi_1'(-1) + C_2\varphi_2'(-1) + C_3\varphi_3'(-1) = 0 \tag{3.2}$$

Here and below a prime denotes the derivative with respect to y. Function φ_1 is the solution of the 'nonviscous' equation

$$(U - c)(\varphi'' - k^2\varphi) - U''\varphi = 0 \tag{3.3}$$

which has no singularity at the point $y = y_c$, where $U(y_0) = c \equiv \omega / k$. For small c we have the equations $\varphi_1(-1) = -c / U'(-1)$, and $\varphi_1'(-1) = 1$ (it is assumed that $U'(y)$ is continuous and the $U'(-1) \neq 0$).

Function φ_2 outside some interval of values of length y of the order $1/(kR)^{1/3}$ with its center at the point y_c coincides with the solution of nonviscous equation (3.3) which has a singularity at the point y_c. Within the indicated interval, φ_2 satisfies the equation which takes viscosity into account.

According to [7], for small c we have

$$\varphi_2(-1) = 1 \qquad {}_2'(-1) = \frac{U''(y_c)}{U'(-1)}\left[f(z) - \ln|kRU'(-1)|^{1/3}\right] \tag{3.4}$$

where $f(z)$ is a complex function of the variable

$$z = c\ [kR\ /\ U''\ (-1)]^{1/3}$$

Here and everywhere below we assume that $-3\pi/2 < \arg k < \pi/2$.

As $z \to \infty$, the function $f(z)$ behaves as $\ln[-zU_1'(-1)]$. Here we take that branch of the logarithm, which is equal to $\ln|z| + i\arg(-z)$, so that for large real $z > 0$ the asymptotic equality $\varphi_2'(-1) = [U''\ (y_c)\ /\ U'(-1)]\ (\ln c - i\pi)$ is fulfilled. This representation is valid [8] if $-\pi/6 < \arg z < 7\pi/6$.

If ω is real and positive, and if $\mathrm{Im}\, k < 0$, then $0 < \arg z < 2\pi/3$. For finite z the function $f(z)$ does not become infinite, since the equation taking account of viscosity which the function φ_2, satisfies, has no singularities [7].

The function φ_3 represents a rapidly varying solution of the 'viscous' equation

$$\varphi'''' = ikR\ (U - c)\,\varphi'' \tag{3.5}$$

and diminishes with increasing y. This solution for large $|kR|$ very rapidly tends to zero with increasing y if $-3\pi/2 < \arg k < \pi/2$.

By introducing the new variable η in place of y, we can transform Equation (3.5) into the form [2, 7 and 9]

$$\frac{d^4\varphi}{d\eta^4} = -\,i\eta\,\frac{d^2\varphi}{d\eta^2} \tag{3.6}$$

where in the neighborhood of the point y_c the variable η is related to y in the following way:

$$\eta = -\,(y - y_c)\ [kRU'\ (-1)]^{1/3}$$

For small values of c, the appropriately normalized solution φ_3 for $y = -1$ satisfies [2, 7 and 9] the conditions

$$\varphi_3(-1) = 1, \quad \varphi_3'(-1) = -\frac{[kRU'\ (-1)]^{1/3}}{D\,(z)} \tag{3.7}$$

Here

$$D\,(z) = \left[\frac{\varphi_3\,(\eta)}{d\varphi_3\,(\eta)\,/\,d\eta}\right]_{\eta=z} \tag{3.8}$$

For large z,

$$D(z) = z^{-1/2}e^{i\pi/4}$$

If the Equation (3.5) we replace $i\omega$ and ik by their complex conjugates, this equation will be satisfied by the solution which is the complex conjugate of the first. On making the indicated substitution, instead of the initial value of z we obtain a value symmetrical to it with respect to the straight line $\arg z = 5\pi/6$.

Since the complex conjugate solution is the same one to within a factor, it is possible to choose the normalization factor in such a way that the solution $\varphi_3(z)$ assumes real values on the straight line $\arg z = 5\pi/6$ and the complex conjugate values at the points symmetrical with respect to this straight line. The same condition must be satisfied by all monomials consisting of the solution $\varphi_3(z)$ and its derivatives invariant relative under the multiplication of φ_3 and z by arbitrary constants, and specifically by the function $D(z)/z$.

The results of the numerical computation of the function φ_3, carried out in [9] confirm the above statements (see the values of φ_3 and its derivatives for $\eta = 0$ and the asymptotic behavior of the function $D(z)$ as $z \to \infty$ cited above).

From the results of [9] it follows that the function $D(z)$ does not vanish or become infinite at finite points of the real axis z, and assumes real values only at the point $z = z_1 \approx 2.3$ and as $z \to \infty$, when $D(z) \to 0$. The aforementioned symmetry property of the function $D(z)/z$ implies that $D(z)$ is likewise bounded on the straight line $\arg z = 2\pi/3$.

The boundary conditions for $y = 1$ are written in the same way. In the case of flows with a symmetrical profile, the conditions for $y = 1$ are usually replaced by the conditions of symmetry of the eigenfunctions in the middle of the channel for $y = 0$. Since the boundary value problem we are considering is invariant relative to the replacement of y by $-y$, all of the eigengunctions are either even or odd. For karge kR, at the point $y = 0$ the contribution of the solution φ_3, to the eigenfunction needs not be considered, since this solution diminishes rapidly with distance from the wall. Hence, in determining the even engenfunctions one must guarantee fulfillment of the condition

$$C_1\varphi_1'(0) + C_2\varphi_2'(0) = 0 \tag{3.9}$$

the analogous condition in determining the odd eigenfunctions being

$$C_1\varphi_1(0) + C_2\varphi_2(0) = 0 \tag{3.10}$$

Together with (3.1) and (3.2), one of these conditions represents a system of equations for finding C_1, C_2, and C_3. Let us denote the linear combinations of the solutions φ_1 and φ_2, which satisfy the conditions (3.9) and (3.10) respectively, by $\varphi^{(1)}$ and $\varphi^{(2)}$. Conditions (3.1) and (3.2) can then be written as

$$\frac{\varphi^{(i)}(-1)}{\varphi^{(i)\prime}(-1)} = \frac{\varphi_3(-1)}{\varphi_3'(-1)} \tag{3.11}$$

4. Let us show first of all that if $k(\omega)$ tends, for real ω, to a real value k_0 as $kR \to \infty$, then $k_0 = 0$ and the corresponding value $c(k_0) \to 0$.

Let us suppose that $k_0 \neq 0$. Here $\arg k_0$ is equal to 0 or $-\pi$, and $\arg z$ is equal to 0 or $2\pi/3$ (we assume that $\omega > 0$).

We consider first the case where c does not tend to zero as $kR \to \infty$. Here $z \to \infty$. Equations (3.7) and (3.8) imply that $[\varphi_3(-1)/\varphi_3'(-1)] \to 0$ as $kR \to \infty$. If c is not small then φ_1 and φ_2 are slowly varying functions of y in the neighborhood of $y = -1$, and Equation (3.11) implies that $\varphi^{(i)}(-1) \to 0$. In the limit we obtain the eigenfunction $\varphi^{(i)}$, which satisfies the nonviscous equation and nonviscous boundary condition of impermeability $\varphi^{(i)}(-1) = 0$. We know [2 and 10], however, that the nonviscous problem for flow with a convex unperturbed velocity profile has a solution corresponding to real k and c only in the case when $k = 0$ and $c = 0$. This contradicts the initial assumtion whereby c does not tend to zero.

Now let $c \to 0$ and $k \to k_0$ as $kR \to \infty$. Since $D(z)$ does not become infinite on the rays $\arg z = 0$ and $\arg z = 2\pi/3$, it follows that the magnitude of $\varphi_3'(-1)$ is of the order of $(kR)^{1/3}$. The quantity $\varphi_2'(-1)$ tends to infinity in the above limiting process, since if z remains finite, then $\varphi_2'(-1) \sim \ln|kR|^{1/3}$, and if $z \to \infty$, then $\varphi_2'(-1) \sim \ln c$.

Here $\varphi_2'(-1)$ does not exceed $\ln (kR)^{1/3}$ in the order of magnitude. The quantity $\varphi_1'(-1)$ remains finite. Hence, $|\varphi_3'(-1) / \varphi_2'(-1)| \to \infty$ and $|\varphi_2'(-1) / \varphi_1'(-1)| \to \infty$ as $kR \to \infty$. Equation (3.2) here implies that either $C_2 \to 0$ and $C_3 \to 0$ (the quantity C_1 is assumed finite) or, that $(C_3/C_2) \to 0$.

In the latter case the term $C_3\varphi_3$ in Equation (3.1) can be neglected.

In $|\varphi_1(-1) / \varphi_2(-1)| \to 0$, Equation (3.1) here implies that $C_2 \to 0$ in this case also, so that $C_3 \to 0$.

Thus, if $c \to 0$ as $kR \to \infty$, solution of the boundary value problem in the limiting case reduces to a nonviscous solution φ_1, which satisfies the nonviscous boundary condition $\varphi_1(-1) = 0$ (since $\varphi_2(-1) \quad c \to 0$). As already stated, this is possible only if $k = 0$ and $c = 0$.

The associated eigenfunction $\varphi_1(y) = U(y)$, is even [10].

From the foregoing it follows that the curves $k^*(\omega)$ which for large kR have points near the real axis of k, correspond to even eigenfunctions and can approach the real axis k only for small k and c.

5. Let us consider the curves $k^*(\omega)$ corresponding to even eigenfunctions for sufficiently large values of kR and small c and k. We shall be interested in the behavior of these curves in the lower half-plane k, so that the inequalities $-2\pi/3 < \arg z < 0$ are fulfilled for the corresponding values of z.

For $k = 0$ the solution φ_1 is of the form [10]

$$\varphi_1(y) = \frac{U(y) - c}{U'(y_c)} \tag{5.1}$$

and is even, so that $\varphi_1'(0) = 0$ for $k = 0$. Equation (3.3) implies that for small values of k the derivative $\varphi_1'(0) = \chi k^2$, where X is a real number. The solutions φ_1 and φ_2 are independent, so that $\varphi_2'(0) \neq 0$ for $k = 0$. Since the derivative $\varphi_2'(0)$ is real ($\varphi_2(y)$ is real [2] for $y > y_c + 0\,[(kR)^{-1/3}]$), it follows that for small k the even nonviscous solution $\varphi^{(1)}$ is of the form

$$\varphi^{(1)} = \varphi_1 + \alpha k^2 \varphi_2 \tag{5.2}$$

where α is a real constant. Using the explicit form [2] of solutions φ_1 and φ_2 in the neighborhood of $y = y_c$ and the constancy of the Wronskian, we can show that $\alpha > 0$. Making use of the values of φ_1, and φ_2 and their derivatives for $y = -1$, we can write Equation (3.11) in the form

$$\frac{c}{\alpha k^2} + c\frac{U''(y_c)}{U'(-1)}\,[f(z) - \ln|kRU'(-1)|^{1/3}] - F(z) = 0 \tag{5.3}$$

Here the first two terms and the function $F(x)$ are equal to

$$[1 + U_1'\varphi^{(1)}(-1) / c\varphi^{(1)\prime}(-1)], \quad F(z) = [1 - D(z) / z]^{-1}$$

respectively.

Equation (5.3) for large z and small real c and k, when $f(z) - \ln (kRU')^{1/3} = \ln c - i\pi$ coincides with the corresponding equation of [2]. As is shown in [6], the fact that the upper half-plane of ω includes part of the unique curve $\omega(k)$ ($\mathrm{Im}\, k = 0$) implies that the function $F(z)$ has neither zeros nor poles in the upper half-plane z. On the real azis $F(z)$

vanishes only for $z = 0$, where it has a simple zero (since $D(0) \neq 0$). Function $F(z) \to 1$ as $z \to \infty$.

If $z \to \infty$ as $kR \to \infty$, then the second term in the left-hand side of Equation (5.3) has the order of $c \ln c$ and tends to zero as $c \to 0$, while $F(z) \to 1$. If z remains finite as $kR \to \infty$, then $F(z)$ is finite, and the second term in (5.3) has the order of $c \ln (kR)^{1/3} \sim c (\ln z - \ln c) \sim c \ln c$ and tends to zero as $c \to 0$. Finally, if $z \to 0$ as $kR \to \infty$, then $F(z) \sim z$, and the second term in Equation (18) is of the order of $[\ln (kR)^{1/3}/(kR)^{1/3}]$ relative to $F(z)$. Thus, the second term in Equation (5.3) for large kR and small c is always small as compared with $F(z)$ and can be neglected in the first approximation. The resulting equation

$$\frac{\omega}{\alpha k^3} = F(z) \tag{5.4}$$

with allowance for the relation $z = c\,[kR \,/\, U''(-1)]^{1/3}$ makes it possible to find the functions $k(z)$ and $\omega(z)$,

$$\alpha R^{1/3} k^{7/3} = \frac{z}{F(z)}, \qquad \alpha R^{3/2} \omega^{7/2} = \frac{z^{9/2}}{F(z)} \tag{5.5}$$

with accuracy directly proportional to kR. One branch $z_*^*(\omega)$ of the curve $z^*(\omega)$ representing the dependence (5.5) of z on real $\omega > 0$, intersects the real axis of z at the point $z = z_1 \approx 2.3$ (where $F(z)$ is real) and with increasing ω goes to infinity in the upper half-plane z, asymptically approaching the real axis [6]. Retention of the second term in Equation (5.3) has the result, that the curve $z_*^*(\omega)$ intersects the real axis a second time at some finite $z = z_2$ such, that the sum of two last terms in Equation (5.3) is real.

The value z_2, as well as z_1, corresponds to real values of ω and k (z_1 and z_2 determine neutral oscillations); $z_2 \to \infty$ as $R \to \infty$, which is a consequence of the asymptotic smallness of the second term in Equation (12) as compared with the third.

Upon introduction of the new variables $\Omega = \alpha^{2/7} R^{3/7} \omega$, and $K = \alpha^{3/7} R^{1/7} k$, Equations (5.5) assume the universal form

$$\Omega^{7/2} = \frac{z^{9/2}}{F(z)}, \qquad K^{7/3} = \frac{z}{F(z)} \tag{5.6}$$

These equations do not include the Reynolds' number or the parameter α, which depends on the form of the unperturbed velocity profile. As $z \to 0$, the quantity Ω tends to zero, and K remains bounded.

On the complex plane K let us consider the curves $K^*(\Omega)$ corresponding to the real values $\Omega > 0$ and lying in the lower half-plane K. The corresponding values of z evidently lie in the sector $0 < \arg z < 2\pi/3$.

As was noted above, $F(z)$ does not vanish or become infinite for $z \neq 0$ in the upper half-plane, hence large value of k on the curves $K^*(\Omega)$ are associated with large values of z, and vice versa. Here $F(z) \approx 1$, and from Equations (5.6), in accordance with Equation (5.4), we obtain

$$\Omega = K^3 \tag{5.7}$$

This equation corresponds to the nonviscous limit in Equation (5.4). By virtue of

Equation (5.7), the upper half-plane Ω on the plane K is associated with three sectors

$$-0 < \arg K < \pi / 3, \quad -2\pi/3 < \arg K < -\pi/3, \quad -4\pi/3 < \arg K < -\pi$$

As follows from (5.6), on the z plane for large z and when $F(z) = 1$, these sectors are associated with the sectors

$$0 < \arg z < 7\pi/3, \quad 4\pi/9 < \arg z < 11\pi/9, \quad 8\pi/9 < \arg z < 5\pi/9$$

each of which for $F(z) = 1$ corresponds to the upper half-plane Ω. We note that all these sectors lie in the region obtained by adding in the upper half-plane z that portion of the plane z, which is symmetrical to it with respect to the straight line $\arg z = 5\pi/6$. Throughout this region, as in the upper half-plane, $F(z) \to 1$ as $z \to \infty$, and the function $F(z)$ has neither zeros nor poles. For not excessively large values of z and k, when $F(z) \neq 1$, the boundaries of the regions corresponding to the upper half-plane Ω on the planes z and K no longer coincide with the boundaries of the indicated sectors.

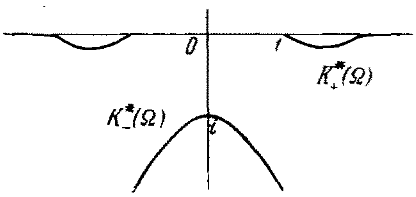

FIG. 1

Let us consider the curve $K^*(\Omega)$ bounding the region Q_-^*, which for large K coincides with the lower sector $-2\pi/3 < \arg K < -\pi/3$. Since $F(z)$ does not vanish on this curve, it follows by (5.6) that $z = 0$ corresponds to the value $\Omega = 0$. For small z we have [9] the equation $F(z) = zA \exp(-i5\pi/6)$, $A \approx 1.1925$. According to the second equation of (5.6), the value $z = 0$ is associated in the lower half-plane K $(-\pi < \arg K < 0)$ only with the value $K = -iA^{-3/7} = -i0.8976$. Making use of Equations (5.6), we can find for this value of K that the argument of the increment dz corresponding to $d\Omega > 0$ is equal to $\pi/3$.

The curve $K_-^*(\Omega)$ and the associated curve $z_-^*(\Omega)$ were obtained with the aid of a computer. To this end we integrated along the line $z_-^*(\Omega)$ the differential equation (3.6), with the initial conditions for $\eta = 0$ taken from [9]. For each z we computed $D(z)$ using Formula (3.8), the right-hand side $\Phi(z) = z^{9/2}[1 - D(z)/z]$ in the first equation of (5.6), and its derivative with respect to z.

The argument of the increment Δz for each step was determined from the condition $\operatorname{Im}[\Phi(z + \Delta z)] = \operatorname{Im}\Phi(z) + \operatorname{Im}[\Phi'(z)\Delta z] = 0$. The argument of the first step was taken equal to $\pi/3$. The length of each step was $|\Delta z| = 0.001$.

Results of the computations appear in Table 1, and the curve $K_-^*(\Omega)$ is shown in Fig. 1. The curve $K_-^*(\Omega)$ corresponding to $\Omega > 0$ lies in the region $\operatorname{Re} K < 0$, $\operatorname{Im} K < 0$ amd represents a portion of the boundary of the region Q^*.

The other portion of the boundary of the region Q^* corresponds to negative values of Ω and is symmetrical to the curve $K_-^*(\Omega)$ with respect to the imaginary axis.

The curve $K_+^*(\Omega)$ representing the boundary of the region Q_+^* (Q_+^* for large K breaks down into the two sectors $0 < \arg K < \pi/3$ and $-4\pi/3 < \arg K < \pi$) was computed in a similar manner in its portion lying in the lower half-plane K. Differential equation (3.6) was computed first along the real axis of z from zero to the point $z = z_1 \approx 2.3$, where $\operatorname{Im}\Phi(z)$

TABLE 1

$n/100$	Ω	Rez	Imz	Re$K_-{}^*$	Im$K_-{}^*$
0	0	0	0	0	—0.8976
2	0.1877	0.08950	0.1787	—0.08085	—0.9061
4	0.3839	0.1601	0.3558	—0.1564	—0.9297
6	0.5937	0.2168	0.5576	—0.2245	—0.9628
8	0.8185	0.2645	0.7518	—0.2855	—1.001
10	1.058	0.3065	0.9473	—0.3405	—1.041
15	1.716	0.3980	1.439	—0.4580	—1.144
20	2.447	0.4806	1.932	—0.5553	—1.244
30	4.080	0.6404	2.919	—0.7102	—1.428
40	5.900	0.8024	3.906	—0.8335	—1.595
60	9.947	1.128	5.879	—1.029	—1.879
80	14.42	1.463	7.851	—1.182	—2.119

vanished, whereupon the argument of each subsequent step Δz was found from the condition Im $\Phi(z + z) = 0$, as was described above. We note that the results of computing differential equation (3.6) along the real axis coincided with the numerical data of [9] to within three places.

The results of computing the curve $K_+^*(\Omega)$ appear in Table 2. The curve $K_+^*(\Omega)$ is shown in Fig. 1. Symmetrical to it with respect to the imaginary axis is the curve corresponding to negative values of Ω and likewise representing the boundary of the region Q_+^*.

TABLE 2

$n/100$	Ω	Rez	Imz	Re$K_+{}^*$	Im $K_+{}^*$
23.5	2.381	2.350	0.01216	1.012	—0.002792
24	2.458	2.387	0.02387	1.037	—0.01548
26	2.818	2.593	0.07980	1.110	—0.04558
30	3.508	2.984	0.1582	1.268	—0.1010
35	4.556	3.478	0.2242	1.487	—0.1474
38	5.239	3.778	0.2410	1.621	—0.1553
40	5.703	3.978	0.2338	1.705	—0.1506
45	6.828	4.474	0.1716	1.880	—0.1082
50	7.837	4.967	0.09227	1.980	—0.05519
55	8.794	5.466	0.05810	2.040	—0.03253
60	9.802	5.966	0.05916	2.106	—0.03157
70	11.99	6.966	0.06851	2.258	—0.03331
80	14.28	7.966	0.06233	2.401	—0.02818

n is the number of steps in computing differential equation (3.6).

From Tables 1 and 2 and Fig. 1 we see that the regions Q_+^* and Q_-^* are separated by a strip parallel to the real axis of K. Hence, Equation (0.4) does not have real roots with Im $\omega > 0$, and plane-parallel flow with symmetrical convex uperturbed velocity profiles for large Reynolds' numbers is not globally unstable.

6. Now let us consider the stability of plane-parallel flow in a pipe when the unperturbed velocity profile has inflection points and when the velocity differs from a constant value only in zones of small width as compared with that of the pipe. These zones

are adjacent to the pipe walls and constitute boundary layers. The unperturbed velocity profile is assumed to be symmetrical and the Reynolds' number is considered sufficiently large.

We shall now assume that the dimensionless quantities y, k and the Reynolds number R in Equation (0.1) are computed from the thickness of the boundary layer (within which $U \neq 1$). Here the dimensionless thickness of the boundary layer is equal to unity, and the dimensionless half-width of the channel can be denoted by $h + 1$ and assumed sufficiently large. We choose the origin of y at the outer boundary of the boundary layer, so that $y = -1$ at the wall and $y = h$ at the center of the channel.

Let us consider for small values of k and c the eigenfunction $\varphi^{(1)}(y)$, which is symmetrical with respect to the middle of the channel. The eigenfunction $\varphi^{(1)}(y)$ can be represented as a linear combination of nonviscous solutions $\varphi^{(1)} = \varphi_1 + A\varphi_2$ everywhere with the exception of the layer adjacent to the wall. The constant A is determined from the condition that $\varphi^{(1)\prime}(h) = 0$ at the center of the channel (see Equation (3.9)).

The boundary conditions at the wall $\varphi(-1) = 0$, and $\varphi'(-1) = 0$ must be satisfied with allowance for the viscous solution φ_3. Just as was done above in obtaining Equation (5.3) through the use of the values of φ_1, φ_2, and φ_3 and their derivatives for $y = -1$, we obtain an equation relating ω and k for an arbitrary A,

$$\frac{\omega}{Ak} + \frac{U''(y_c)}{U'(-1)} c\,[f(z) - \ln|kRU'(-1)|^{1/3}] - F(z) = 0 \tag{6.1}$$

In Section 5 we showed that if $k \to 0$ with a constant unperturbed velocity profile, then $A \to \alpha k^2$, and this value was used in deriving Equation (5.3). If, on the other hand, $h \to \infty$ as $k \to 0$ in such a way that the product kh does not tend to zero, then, as will be shown below, the quantity A does not tend to the value αk^2.

Outside the boundary layer $U''(y) = 0$ and any nonviscous solution can be represented as the linear combination $C_1 e^{ky} + C_2 e^{-ky}$. For $k = 0$ the solution φ_1 is proportional to $U(y) - c$, so that $\varphi_1'(0) = 0$. For small values of k the quantity $\varphi_1'(0)$ is of the order of k^2. Neglecting this quantity, we find, that for $y > 0$

$$\varphi_1 = \cosh ky \tag{6.2}$$

Let us denote the values of $\varphi_2(0)$ and $\varphi_2'(0)$ for $k = 0$ and $c = 0$ by a and b (a and b are real), respectively. From the linear independence of the solutions φ_1 and φ_2 it follows that $b \neq 0$. For small k and c the values $\varphi_2(0)$ and $\varphi_2'(0)$ can differ from a and b, respectively, by not more than a quantity of the order of magnitude of max (k^2, c). This implies that the equation

$$\varphi_2 = \frac{b}{k} \sinh ky \tag{6.3}$$

is approximately fulfilled for $y > 0$ in the case of small k and c.

From the condition $\varphi'(h) = 0$ we find here that

$$A = -\frac{k}{b} \tanh kh \tag{6.4}$$

If k is decreased without limit for a fixed h, then A becomes proportional to k^2, as was assumed in the derivation of (5.3). If, on the other hand, h tends to infinity for a fixed

k and Re $k > 0$, then the first term in Equation (5.1) assumes the form ω/k^2 chacteristic of a boundary layer [2 and 11]. As shown above, for small k and c and a sufficiently large kR, the second term in Equation (5.1) is small as compared with $F(z)$ and can be neglected in the first approximation. Equation (5.1) then becomes

$$\frac{b\omega}{k^2 \tanh kh} = F(z) \tag{6.5}$$

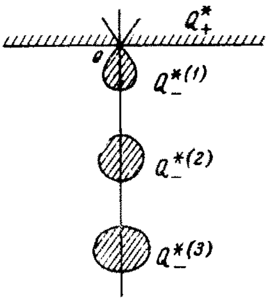

FIG. 2

Let us first consider the nonviscous case when $F(z) = 1$. Here the regions Q_+^* and Q_-^* are of the form shown in Fig. 2. Each of the closed regions $Q_-^{*(1)}$, $Q_-^{*(2)}$, . . . in the lower half-plane k corresponds to the entire upper half-plane ω. The size of each of them along

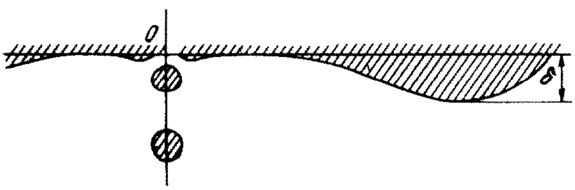

FIG. 3

the vertical is $\pi/2h$. For a sufficiently large Reynolds number, the effect of the viscosity on the picture in Fig. 2 is limited to regions where the quantity z, whose order of magnitude is $\omega R^{1/3}/k^{2/3}$, is small, i.e. to neighborhoods of the points $k = 0 \pm i\pi n/h$ at which ω vanishes. Specifically, as was shown above, in the small neighborhood $|k| \ll 1/h$ of the point $k = 0$, the behavior of the curves bounding the regions Q_+^* and Q_-^* is of the form depicted in Fig. 1.

Now let us consider somewhat larger values of k when one must consider terms in the dispersion equation in addition to those retained in Equation (6.5). We shall assume that $F(z) = 1$ and Re $kh \gg 1$, so that $kh = 1$ for k with Re $k > 0$. Here the dispersion equation of the problem under consideration coincides with the dispersion equation for a nonviscous boundary layer. As we know [2 and 10], if the unperturbed velocity profile contains an inflection point, then there exists an interval on the real axis of k whose points are associated with ω whose Im $\omega > 0$. This means that in this case the boundary of the region Q_+^* which occupies a large part of the upper half-plane k, drops below the real axis of k, as is shown in Fig. 3. The depth δ by which the lower boundary of the region Q_+^* drops tends to a finite limit as $h \to \infty$. If h is sufficiently large, so that $\pi/2h < \delta$, it follows that lower boundary of the region $Q_-^{*(1)}$ turns out to lie above the lower point of the region Q_+^*. Since the region $Q_-^{*(1)}$ corresponds to the entire upper half-plane ω, it is clear that there must exist pairs of values of k, one of which belongs to the region Q_+^* and the other to $Q_-^{*(1)}$ corresponding to the same ω with Im $\omega > 0$. Thus, Equation (0.4) is fulfilled for certain values of ω from the upper half-plane, and the flow under consideration is globally unstable.

The above result appears to indicate that the boundary layer is unstable at sufficiently large Reynolds' numbers, although on taking the limit as $h \to \infty$ in the nonviscous case the dimensions of the regions $Q_+^{*(1)}$, $Q_-^{*(2)}$, ... situated in the lower half-plane diminish without limit, and each of them tends to the point $k = 0$.

In the presence of viscosity and a finite h, boundaries of the regions $Q_+^{*(1)}$, $Q^{*2)}$ . . ., alter their positions relative to the nonviscous case, although in the neighborhoods of the points where ω, and therefore z, become infinite, the function $k(\omega)$ retains the same form as it has in the nonviscous case.

Perturbations corresponding to the value of $k(\omega)$ from the region $Q_-^{*(1)}$ for large h are gradually damped out with distance from the wall (the damping decrement is equal to Re $k(\omega)$). The perturbations in question propagate principally in the outer region with repect to the boundary layer and, as is easy to verify, represent solutions of the Laplace equation.

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